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One-Vortex Moduli Space and Ricci Flow

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Abstract

The metric on the moduli space of one abelian Higgs vortex on a surface has a natural geometrical evolution as the Bradlow parameter, which determines the vortex size, varies. It is shown by various arguments, and by calculations in special cases, that this geometrical flow has many similarities to Ricci flow.

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1 Introduction

In the abelian Higgs model at critical coupling, defined in the plane, there are static N-vortex solutions in which the Higgs field vanishes at precisely N (not necessarily distinct) points. The moduli space of solutions $\mathcal{M}_N$ is a manifold of complex dimension N. There is a natural Kähler metric on $\mathcal{M}_N$, and motion along a geodesic in the moduli space corresponds to an N-vortex motion at slow speeds, approximating a solution of the time-dependent field equations [14, 21].

The abelian Higgs model can be straightforwardly extended to any smooth surface Σ without boundary, for example the hyperbolic plane, or a compact Riemann surface. A metric on Σ must be specified. Provided the area of Σ is sufficiently large, there is again a moduli space of N-vortex solutions, with a natural Kähler metric.

In this paper we shall investigate the case of one vortex on a compact surface Σ with arbitrary metric. Little attention has previously been paid to this apparently simple case. It is helpful here to use Bradlow's formulation of the vortex equations. Bradlow introduced an explicit positive parameter τ such that the area of a vortex is proportional to τ [3][1]. In some of the literature, τ is scaled to unity and the area A of Σ regarded as variable, with the ratio of A to the area of a vortex the physically interesting quantity. However we shall not do this here. We shall consider the area A as fixed, but τ as variable. 1-vortex solutions exist provided $0 < \tau < \frac{A}{4\pi}$. The 1-vortex moduli space $\mathcal{M}_1$ we shall simply denote by $\mathcal{M}$ from now on. As a manifold, $\mathcal{M}$ is a copy of Σ, since given any point of Σ, there is a unique vortex solution whose Higgs field vanishes at that point. The metric on $\mathcal{M}$ depends on τ and generally differs from the metric on Σ. For example, for a vortex on a round sphere, $\mathcal{M}$ is also a round sphere, by symmetry, but its area is smaller. For a vortex on a flat torus, the moduli space is the same flat torus.

Our intuition is that in the limit $\tau \to 0$, the metric on moduli space will be the original metric on Σ, since a vortex is pointlike in this limit, and should move along geodesics of Σ. Then, as τ increases, the moduli space metric will partly smooth out the wrinkles in the metric of Σ. This is because the vortex occupies a finite region of Σ. Its motion, according to the geodesic approximation, is along a geodesic of $\mathcal{M}$, so the metric on $\mathcal{M}$ should depend on the metric on Σ averaged in some way over the region of the vortex. We shall study how the metric on $\mathcal{M}$ varies with τ, and think of this as a geometrical flow, with τ regarded as an analogue of time.

Calculating the metric on moduli space is not possible explicitly, in general, but there are a number of mathematical results which determine some of its properties. Samols found a formula for the metric on $\mathcal{M}_N$ in terms of coefficients in the expansion of the Higgs field around each vortex centre [19]. From this formula it follows that the metric is Kähler. For vortices on a compact Riemann surface Σ, the real cohomology class of the Kähler 2-form on $\mathcal{M}_N$ is known, and from this the volume of $\mathcal{M}_N$ can be computed [16]. It depends on the area of

[1] In our convention, τ is the inverse of the parameter that appears in [3].

Σ, the genus of Σ, and on τ and N. These results simplify in the 1-vortex case.

In this paper, we give the area of the 1-vortex moduli space $\mathcal{M}$ as a function of τ, and show that the flow of the area with τ coincides with what occurs in Ricci flow. We argue that for small τ, the metric itself on $\mathcal{M}$ evolves from the metric on Σ by Ricci flow. As is well-known [13], Ricci flow tends to smooth out the metric. For larger values of τ we cannot compute the metric on $\mathcal{M}$ in general. However, for the case of a vortex on a slightly deformed sphere of area 4π, we calculate the asymptotic metric on $\mathcal{M}$ for τ approaching 1. The result is a round collapsing sphere, agreeing with what occurs in Ricci flow. For one vortex on a torus of area 4π we show that no matter what the starting metric on the torus, when τ reaches 1, $\mathcal{M}$ is a flat torus. This is similar to the result of Ricci flow, except that Ricci flow takes an infinite time to produce a flat torus. These results suggest that in some precise way, the geometric flow on the moduli space of one vortex is related to Ricci flow, but they are not conclusive. It is an open problem to obtain a general differential equation for the geometrical flow of the moduli space metric, and see whether or not it is equivalent to Ricci flow.

We believe our results are of interest from the point of view of connecting Ricci flow to the issue of the motion of finite-sized objects in gravity [4]. A basic axiom in gravity is that pointlike particles move along geodesics of space-time. Our vortices, being solitons, behave like particles, and when they are vanishingly small, and moving slowly, they move along geodesics of Σ. However, as τ increases from zero, the vortex size increases, and vortex motion is along a geodesic in the moduli space $\mathcal{M}$, which for one vortex is still a path in Σ, but with a metric modified by Ricci flow. This gives a physical interpretation of two-dimensional Ricci flow, at least for small times. It generates effective metrics for particles of small, finite size, whose geodesics are the approximate trajectories of these particles. This is a more classical physical interpretation of Ricci flow than the known interpretation involving the renormalisation group flow of sigma models [6]. For a recent discussion of possible physical interpretations of geometrical flows, see ref.[12].

2 Vortex on a Surface

Let Σ be a compact Riemann surface of genus g with local complex coordinate $z = x + iy$. We assume that Σ has a metric compatible with the complex structure

$$ds^2 = \Omega(x,y)(dx^2 + dy^2) = \Omega(x,y)dzd\bar{z}\,. \tag{2.1}$$

Ω is called the conformal factor. The fields of the abelian Higgs model on Σ are locally a complex scalar field ϕ and an abelian gauge potential whose components (a_x, a_y) combine naturally into a 1-form $a = a_x dx + a_y dy$. Globally, these are a section and $U(1)$ connection of a line bundle over Σ. The magnetic field is $B = \partial_x a_y - \partial_y a_x$ and the first Chern number of the bundle is

$$c_1 = \frac{1}{2\pi}\int_\Sigma B\, d^2x\,, \tag{2.2}$$

which is an integer that can be identified as both the number of magnetic flux quanta and the net number of vortices on Σ.

This abelian Higgs model extends to a dynamical field theory on the space-time $\Sigma \times \mathbb{R}$, with its product metric. We shall consider the case where the scale parameters of the model are at "critical coupling". For a discussion of the Lagrangian at critical coupling, including its kinetic and potential energy terms, see ref. [17]. Static solutions which minimize the potential energy satisfy the first order Bogomolny equations [2]

$$\begin{aligned} D_x\phi + iD_y\phi &= 0\,, & (2.3)\\ \frac{1}{\Omega}B - \frac{1}{2}\left(\frac{1}{\tau} - |\phi|^2\right) &= 0\,, & (2.4) \end{aligned}$$

where $D_i\phi = \partial_i\phi - ia_i\phi$, and τ is the positive (constant) Bradlow parameter.

The pair of Bogomolny equations can be reduced to a single gauge invariant equation as follows [11, 22]. With $z = x + iy$ and $\bar{z} = x - iy$ we have $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Let $a_z = \frac{1}{2}(a_x - ia_y)$ and $a_{\bar{z}} = \frac{1}{2}(a_x + ia_y)$. The first Bogomolny equation becomes $\partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0$, whose solution is

$$a_{\bar{z}} = -i\partial_{\bar{z}}(\log\phi)\,. \tag{2.5}$$

a_z is the complex conjugate of this. Now set

$$\phi = e^{\frac{1}{2}h + i\chi} \tag{2.6}$$

where h is the gauge invariant and globally well-defined quantity $\log|\phi^2|$. Then $B = -2i(\partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z) = -2\partial_z\partial_{\bar{z}}h$. Substituting into the second Bogomolny equation we obtain, for one vortex, Taubes' equation in the form

$$\Delta h + \frac{1}{\tau} - e^h = \frac{4\pi}{\Omega}\delta^{(2)}(\mathbf{x} - \mathbf{X})\,. \tag{2.7}$$

$\Delta h \equiv \frac{4}{\Omega}\partial_z\partial_{\bar{z}}h$ is the covariant Laplacian of h on Σ. The delta function source arises from the logarithmic singularity of h at the vortex centre $\mathbf{X}$, where ϕ vanishes and the magnetic flux density has its maximal value, $\frac{1}{2\tau}$. We denote by Z the complex coordinate of the vortex centre, $Z = X + iY$.

The key constraint on the existence of vortices arises by integrating the second Bogomolny equation over Σ (with the geometrical measure $\Omega\, d^2x$), or equivalently, by integrating (2.7) (taking care over the logarithmic singularity). One finds, for one vortex,

$$\frac{A}{2\tau} - \frac{1}{2}\int_\Sigma |\phi^2|\Omega\, d^2x = 2\pi\,, \tag{2.8}$$

and since $|\phi^2|$ and Ω are non-negative, there is the Bradlow inequality $\tau \leq \frac{A}{4\pi}$. When τ approaches $\frac{A}{4\pi}$, the Higgs field becomes small everywhere and the magnetic flux density $\frac{1}{\Omega}B$ is almost uniform, that is, B is approximately equal to $\frac{2\pi\Omega}{A}$. The vortex dissolves in the limit. The Bogomolny equations have solutions

with uniform magnetic flux density for $\tau = \frac{A}{4\pi}$, but ϕ vanishes identically, so the solutions are not vortices. Therefore, 1-vortex solutions of the Bogomolny equations exist only in the range

$$0 < \tau < \frac{A}{4\pi} \,. \tag{2.9}$$

(Similarly, N-vortex solutions exist only for $0 < \tau < \frac{A}{4\pi N}$.) It is a result of Bradlow [3] and García-Prada [7] that for τ in this range, there is a unique vortex solution for each choice of Z on Σ.

It follows from the Lagrangian and Bogomolny argument (see [17]) that the energy of one vortex is $E = \frac{\pi}{\tau}$. This can also be interpreted as the vortex mass. The size (that is, area) of a vortex is of order $4\pi\tau$. This estimate comes from treating the magnetic flux density as approximately $\frac{1}{2\tau}$ in the vortex core and zero outside (really, it decays exponentially), and recalling that the total flux is 2π. The Bradlow inequality arises, intuitively, because a vortex of size $4\pi\tau$ cannot be fitted into an area A smaller than this.

3 The Vortex Moduli Space and its Metric

Provided τ satisfies the inequality (2.9), there is a moduli space of 1-vortex solutions $\mathcal{M}$, and as a manifold $\mathcal{M} = \Sigma$. There is a natural metric on the moduli space. Mathematically, it is the L^2 norm on fields tangent to the moduli space which are also orthogonal to infinitesimal gauge transformations. Physically, it is derived from the expression for the kinetic energy of a vortex with slowly moving centre, $Z(t)$. Samols obtained a formula for the metric in terms of the local expansion of the field $h = \log|\phi|^2$ about the vortex centre [19]. When the vortex is at Z, $h - 2\log|z - Z|$ is a real, regular function, so h has an expansion

$$h(z, \bar{z}) = 2\log|z - Z| + a + \frac{1}{2}\bar{b}(z - Z) + \frac{1}{2}b(\bar{z} - \bar{Z}) + \cdots \,, \tag{3.1}$$

where a and b are functions of Z, $\bar{Z}$ and τ. The kinetic energy of the moving vortex is then

$$T = \frac{1}{2}\frac{\pi}{\tau}\left(\Omega + 2\tau\frac{\partial b}{\partial Z}\right)\dot{Z}\dot{\bar{Z}} \tag{3.2}$$

with Ω evaluated at Z. Dropping the mass factor $\frac{\pi}{\tau}$ and the factor $\frac{1}{2}$, we obtain the metric on $\mathcal{M}$,

$$ds^2\big|_{\mathcal{M}} = \left(\Omega + 2\tau\frac{\partial b}{\partial Z}\right) dZ d\bar{Z} \,, \tag{3.3}$$

which, though defined using the local coordinate Z, can be shown to be globally consistent. This metric is a deformation of the original metric on Σ, the deformation being small when τ is small, as we will see below. The formula shows that globally, for all τ, the metric on $\mathcal{M}$ is in the same conformal class as the metric on Σ.

It is not easy to determine b in most situations, so, in general, the metric on $\mathcal{M}$ is not known. Some remarkable topological information about b is however known, which we briefly review, following [16]. Note first that b is not invariant under changes of coordinate. If, locally, we use a different complex coordinate chart $z' = z'(z)$ on Σ, then h has a similar expansion around Z' as (3.1),

$$h(z', \bar{z}') = 2 \log |z' - Z'| + a' + \frac{1}{2}\bar{b}'(z' - Z') + \frac{1}{2}b'(\bar{z}' - \bar{Z}') + \cdots , \tag{3.4}$$

where, because of the logarithmic term, $a' = a - 2 \log |\frac{\partial z'}{\partial z}|$ and

$$\bar{b}' = \left(\frac{\partial z'}{\partial z}\right)^{-1} \bar{b} - \left(\frac{\partial z'}{\partial z}\right)^{-2} \frac{\partial^2 z'}{\partial z^2} , \tag{3.5}$$

with the quantities on the right hand side evaluated at Z. Let us define the 1-form $-\bar{b}(Z, \bar{Z})\, dZ$, and its transformed version $-\bar{b}'(Z', \bar{Z}')\, dZ'$. Then

$$-\bar{b}'\, dZ' = -\bar{b}\, dZ + \left(\frac{\partial Z'}{\partial Z}\right)^{-1} \frac{\partial^2 Z'}{\partial Z^2}\, dZ . \tag{3.6}$$

This is exactly the same transformation rule as for a connection 1-form on the holomorphic cotangent bundle of Σ, the canonical bundle K, whose sections are locally $f(Z, \bar{Z})\, dZ$ and where the transition rule is $f' = (\frac{\partial Z'}{\partial Z})^{-1} f$. Globally therefore, $-\bar{b}\, dZ$ is a connection 1-form on K, varying with τ. The Chern form of this connection is

$$C_1(K) = \frac{i}{2\pi} d(-\bar{b}\, dZ) = \frac{i}{2\pi} \left(\frac{\partial \bar{b}}{\partial \bar{Z}}\right) dZ \wedge d\bar{Z} . \tag{3.7}$$

For Σ compact and of genus g, the integral of C_1 over Σ is the Chern number, $c_1(K) = 2(g - 1)$ [10].

The Kähler 2-form associated to the metric (3.3) is

$$\omega = \frac{i}{2} \left(\Omega + 2\tau \frac{\partial \bar{b}}{\partial \bar{Z}}\right) dZ \wedge d\bar{Z} , \tag{3.8}$$

where we have used the reality property

$$\frac{\partial b}{\partial Z} = \frac{\partial \bar{b}}{\partial \bar{Z}} \tag{3.9}$$

proved in [17]. Integrating ω over Σ gives the area of the 1-vortex moduli space $\mathcal{M}$,

$$A|_{\mathcal{M}} = A + 2\pi\tau c_1(K) = A + 4\pi\tau(g - 1) , \tag{3.10}$$

where A is the area of Σ. For all g, the range of the Bradlow parameter τ is as given in eq.(2.9). The range of $A|_{\mathcal{M}}$ is therefore from A to gA. When $g = 0$, the area of $\mathcal{M}$ vanishes as $\tau \to \frac{A}{4\pi}$ whereas for $g = 1$ the area remains constant.

For a vortex on a sphere, plane or hyperbolic plane, in each case with its standard metric of constant curvature, the metric on $\mathcal{M}$ can be computed explicitly, and is [15, 19]

$$ds^2\big|_{\mathcal{M}} = \left(1 - \frac{\tau R_0}{2}\right)\Omega\, dZ d\bar{Z}\,, \tag{3.11}$$

a rescaled version of the metric on the underlying surface Σ. Here R_0 is the Ricci scalar curvature, which for a sphere of radius r is $\frac{2}{r^2}$. The result for the plane, with $R_0 = 0$, extends to any flat torus, but it is not known if the result for the hyperbolic plane extends to a compact surface of constant negative curvature. The metric (3.11) has been obtained using a symmetry argument to find b, rather than solving the Taubes equation for h, but in the case when $R_0 = -\frac{1}{\tau}$, Taubes' equation reduces to Liouville's equation, and the vortex solution and b have a simple algebraic form, leading explicitly to $ds^2\big|_{\mathcal{M}} = \frac{3}{2}\Omega\, dZ d\bar{Z}$ [20].

4 Vortex Moduli space for small τ, and Ricci flow

Consider, as before, one vortex on a general surface Σ with smooth metric $ds^2 = \Omega\, dz d\bar{z}$. We give a simple argument which determines the metric on the moduli space $\mathcal{M}$ when τ is small. In this regime, the vortex is small, since its magnetic flux is concentrated in a region of area $4\pi\tau$, and outside this region the Higgs field is everywhere close to its constant vacuum value, $|\phi|^2 = \frac{1}{\tau}$. The vortex therefore only detects the local intrinsic curvature of Σ at the vortex centre Z, and it will move with kinetic energy equal to what the kinetic energy would be on a surface of constant curvature. Using eq.(3.11) we deduce that the metric on $\mathcal{M}$ is

$$ds^2\big|_{\mathcal{M}} = \left(1 - \frac{\tau R}{2}\right)\Omega\, dZ d\bar{Z}\,. \tag{4.1}$$

Here, R is the (non-uniform) Ricci scalar curvature evaluated at Z, for which the formula is $R = -\frac{4}{\Omega}\partial_Z\partial_{\bar{Z}}(\log\Omega)$. In the limit $\tau \to 0$, the metric on $\mathcal{M}$ smoothly becomes the original metric on Σ, so geodesics on $\mathcal{M}$ become geodesics on Σ, the result one expects for pointlike particles.

We do not claim any rigour for this result, and corrections to the metric of order τ^2 are expected. A more careful asymptotic analysis of the vortex solutions on Σ as $\tau \to 0$ would be needed to prove it. The result is compatible with the exact formula (3.10) for the area of $\mathcal{M}$, in the case that Σ is compact, because the Gauss–Bonnet formula,

$$\frac{i}{2}\int_\Sigma R\Omega\, dZ \wedge d\bar{Z} = 8\pi(1-g)\,, \tag{4.2}$$

implies that the area of $\mathcal{M}$ with the approximate metric (4.1) is $A + 4\pi\tau(g-1)$.

Let us now write the exact metric on $\mathcal{M}$ as

$$ds^2\big|_{\mathcal{M}} = \Omega(\tau)\, dZ d\bar{Z}\,, \tag{4.3}$$

where $\Omega(0) = \Omega$. The expression (4.1) can be interpreted as saying that for small τ, $\Omega(\tau)$ evolves by Ricci flow. This is because the Ricci flow equation on the surface Σ (with complex coordinate Z, and "time" t) is [13]

$$\frac{\partial}{\partial t}\Omega(t) = -\frac{1}{2}R(t)\Omega(t)\,, \tag{4.4}$$

where $R(t)$ is the Ricci scalar curvature of the metric with conformal factor $\Omega(t)$, and the dependence of Ω and R on Z and $\bar{Z}$ is implied. The initial condition is also $\Omega(0) = \Omega$. The short-time solution of (4.4) is

$$\Omega(t) = \left(1 - \frac{tR(0)}{2}\right)\Omega\,, \tag{4.5}$$

and this agrees with (4.1) if we identify τ with the time t of the Ricci flow.

The area of the metric on Σ under Ricci flow is $A + 4\pi t(g-1)$, which is derived from (4.4) by integrating over Σ and using the Gauss–Bonnet formula. This agrees with the area of $\mathcal{M}$ for all τ, if τ and t are identified. It would be very interesting if $\Omega(\tau)$ were precisely the solution $\Omega(t)$ of the Ricci flow, with initial condition Ω. To investigate this further, we shall consider the other limit, where $\tau \to \frac{A}{4\pi}$ and the magnetic flux of the vortex is spread almost uniformly over the surface Σ. All our results scale in rather a simple way if the area A and Bradlow parameter τ are multiplied by the same factor. So let us from now on fix $A = 4\pi$, which implies that the range of τ is $0 < \tau < 1$.

5 Linearized Ricci Flow on a Sphere

We need to review some results for the large-time asymptotics of two-dimensional Ricci flow. In this section we consider Σ a compact surface of genus zero, that is, Σ is conformally a 2-sphere. The metric on Σ can be written as

$$ds^2 = e^u \Omega_0 \, dz d\bar{z}\,, \tag{5.1}$$

where z is a stereographic coordinate and $\Omega_0 = 4/(1+z\bar{z})^2$ defines the round metric on a sphere of unit radius. $e^{u(z,\bar{z})}$ is the globally-defined conformal factor relative to the round sphere, so it should have a finite limit as $z \to \infty$. For the sphere to have area 4π we require

$$\frac{i}{2}\int e^u \Omega_0 \, dz \wedge d\bar{z} = 4\pi\,. \tag{5.2}$$

Let us now assume that u is small, and consider the Ricci flow of this metric. We first find the normalised Ricci flow, with time variable $\tilde{t}$, and then rescale the metric and time to obtain the true Ricci flow.

On a surface of genus zero and area 4π the averaged Ricci scalar curvature is 2. The normalised Ricci flow equation is therefore

$$\frac{\partial \widetilde{\Omega}}{\partial \tilde{t}} = -\frac{1}{2}(\widetilde{R} - 2)\widetilde{\Omega}\,, \tag{5.3}$$

where $\widetilde{R}$ is the Ricci curvature for the metric $\widetilde{\Omega}\, dzd\bar{z}$. The round sphere with conformal factor Ω_0 is a fixed point, so let us consider the linearized equation for metrics close by. Set

$$\widetilde{\Omega} = e^u \Omega_0 \,. \tag{5.4}$$

The (exact) Ricci curvature is

$$\widetilde{R} = e^{-u}(-\Delta_0 u + R_0) \,, \tag{5.5}$$

where $R_0 = 2$ is the scalar curvature for Ω_0 and Δ_0 is the covariant Laplacian operator in the Ω_0 background. To linear order in u, $\widetilde{R} = 2 - 2u - \Delta_0 u$, so the normalised Ricci flow equation is

$$\frac{\partial u}{\partial \tilde{t}} = u + \frac{1}{2}\Delta_0 u \,. \tag{5.6}$$

u can be expanded in spherical harmonics, and for a harmonic Y_j with angular momentum j, Δ_0 has eigenvalue $-j(j+1)$. The coefficient c_j of such a harmonic therefore satisfies

$$\frac{dc_j}{d\tilde{t}} = \left(1 - \frac{1}{2}j(j+1)\right) c_j \,, \tag{5.7}$$

so it generally decays exponentially with increasing $\tilde{t}$. The coefficient c_0 must vanish because of our assumption that the area of the deformed sphere is 4π. The coefficient c_1 can also be chosen to vanish, as a non-zero value just corresponds to a coordinate reparametrization of the sphere (since the curvature remains constant). The non-trivial coefficient that decays most slowly is c_2, so generic deformed spheres approach the round sphere at late times with a deformation that is a $j = 2$ spherical harmonic Y_2 (some real, linear combination of the five standard harmonics $Y_{2m} : -2 \leq m \leq 2$). If the initial metric is a round sphere with just a small deformation by a $j = 2$ harmonic, $\widetilde{\Omega}(0) = (1 + Y_2)\Omega_0$, then the normalised Ricci flow gives

$$\widetilde{\Omega}(\tilde{t}) = (1 + e^{-2\tilde{t}} Y_2)\Omega_0 \,. \tag{5.8}$$

In two dimensions, Ricci flow is related to the normalised Ricci flow by a simple scaling of the metric and a reparametrisation of time, as follows. Suppose solutions of eqs.(4.4) and (5.3) are related by

$$\Omega = \mu \widetilde{\Omega} \,, \tag{5.9}$$

where μ depends only on time. Then $R = \frac{1}{\mu}\widetilde{R}$. Substituting in (4.4), one obtains

$$\frac{\partial \widetilde{\Omega}}{\partial t} = -\frac{1}{\mu}\frac{d\mu}{dt}\widetilde{\Omega} - \frac{1}{2\mu}\widetilde{R}\widetilde{\Omega} \,, \tag{5.10}$$

and hence

$$\frac{\partial \widetilde{\Omega}}{\partial \tilde{t}} = -\frac{1}{\mu}\frac{d\mu}{d\tilde{t}}\widetilde{\Omega} - \frac{1}{2\mu}\frac{dt}{d\tilde{t}}\widetilde{R}\widetilde{\Omega} \,. \tag{5.11}$$

This agrees with eq.(5.3) provided

$$\frac{dt}{d\tilde{t}} = \mu \qquad \text{and} \qquad -\frac{1}{\mu}\frac{d\mu}{d\tilde{t}} = 1\,. \tag{5.12}$$

The joint solution, assuming as initial condition that $\widetilde{\Omega} = \Omega$ at, respectively, $\tilde{t} = 0$ and $t = 0$, is

$$\mu = e^{-\tilde{t}}\,, \quad t = 1 - e^{-\tilde{t}}\,. \tag{5.13}$$

As $\tilde{t}$ runs from 0 to ∞, t runs from 0 to 1. Applying this to the linearized solution of the normalised Ricci flow (5.8), we obtain the corresponding solution of the true Ricci flow

$$\Omega(t) = (1-t)(1 + (1-t)^2 Y_2)\Omega_0\,. \tag{5.14}$$

As t approaches 1, the area of the sphere approaches zero linearly with t, and the deformation relative to a round sphere approaches zero quadratically.

We shall next study the metric on moduli space for one vortex on a slightly deformed sphere, with deformation a $j = 2$ harmonic Y_2. If the geometrical flow of the moduli space metric were the same as Ricci flow, then the conformal factor on $\mathcal{M}$ would be

$$\Omega(\tau) = (1-\tau)(1 + (1-\tau)^2 Y_2)\Omega_0\,. \tag{5.15}$$

We shall show that as the Bradlow parameter τ approaches 1, the area of moduli space behaves as in (5.15) and that the deformation of the sphere does decay to zero at least linearly. But our calculation is not refined enough to show that the deformation is proportional to $(1-\tau)^2$.

6 Vortex on a deformed sphere

Explicit calculation of the metric on the 1-vortex moduli space $\mathcal{M}$ is generally hard, since the vortex solution is not explicitly known even for a vortex on a plane. For a vortex on a round sphere the metric on $\mathcal{M}$ is known, but for a vortex on a deformed sphere it is not. Here we shall assume the deformation is by the simplest $j = 2$ harmonic and small, and we shall work to linear order in the deformation. We still cannot calculate the metric on $\mathcal{M}$ for the whole range of the Bradlow parameter τ, so we concentrate on the case where τ is close to 1, complementing our earlier results where it was close to 0.

To explain our strategy, we first rederive the metric on moduli space for the case of a vortex on the round unit sphere with τ close to 1 [15, 1]. It will be convenient to switch frequently between polar coordinates θ, φ and the stereographic coordinate $z = \tan\frac{1}{2}\theta\, e^{i\varphi}$. We start with Taubes' equation

$$\Delta_0 h + \frac{1}{\tau} - e^h = \frac{4\pi}{\Omega_0}\delta^{(2)}(\mathbf{x} - \mathbf{X}) \tag{6.1}$$

where $\Delta_0 = \frac{4}{\Omega_0}\partial_z\partial_{\bar{z}} = (1 + z\bar{z})^2\partial_z\partial_{\bar{z}}$ is the covariant Laplacian on the unit sphere, and we set $\frac{1}{\tau} = 1 + \epsilon$. For small ϵ, e^h is small everywhere, of order ϵ,

and vanishes at the vortex centre. A consistent expansion is to set

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \cdots \tag{6.2}$$

where h_0 has a summand $\log \epsilon$. h_0 satisfies

$$\Delta_0 h_0 + 1 = \frac{4\pi}{\Omega_0} \delta^{(2)}(\mathbf{x} - \mathbf{X}) \,. \tag{6.3}$$

For a vortex at the origin, $\mathbf{X} = \mathbf{0}$, the solution is

$$h_0 = \log \frac{z\bar{z}}{1 + z\bar{z}} + \log C_0 \epsilon \,, \tag{6.4}$$

where the constant C_0 is still to be determined. Expanding out Taubes' equation up to terms of order ϵ we find the inhomogeneous equation for h_1,

$$-\Delta_0 h_1 = 1 - C_0 \frac{z\bar{z}}{1 + z\bar{z}} \,. \tag{6.5}$$

In spherical polars, the operator on the left hand side is (minus) the standard Laplacian on a unit 2-sphere, and

$$\frac{z\bar{z}}{1 + z\bar{z}} = \frac{1}{2}(1 - \cos\theta) \,. \tag{6.6}$$

The right hand side of (6.5) is required to have an expansion in spherical harmonics with no constant term, and this fixes $C_0 = 2$. The right hand side is then $\cos\theta$, a $j = 1$ spherical harmonic, so $h_1 = \frac{1}{2}\cos\theta + C_1$. Expanding to next order in ϵ we get an equation for h_2 which imposes a consistency condition on C_1. Although we do not actually need these, we find $C_1 = \frac{1}{6}$ and h_2 is a linear combination of the Legendre polynomials $P_2(\cos\theta)$ and $P_1(\cos\theta)$ and an undetermined constant C_2. Proceeding further one could construct a formal series expansion for h in powers of ϵ, where each term is a finite polynomial in $\cos\theta$ or equivalently a finite sum of Legendre polynomials $P_m(\cos\theta)$. Converting back to a function of z and $\bar{z}$, our solution for h, up to order ϵ, is

$$h = \log \frac{z\bar{z}}{1 + z\bar{z}} + \log 2\epsilon + \epsilon \left(\frac{2}{3} - \frac{z\bar{z}}{1 + z\bar{z}} \right) + \cdots \,. \tag{6.7}$$

Spherical symmetry means that it is easy to convert this solution into the solution for a vortex located at a general point $\mathbf{X}$ on the sphere. Let Z be the stereographic coordinate corresponding to $\mathbf{X}$. We observe that $\frac{4z\bar{z}}{1+z\bar{z}}$ is the square of the chordal distance from z to 0 on the unit sphere, and the rotated version of this function is

$$\frac{4(z - Z)(\bar{z} - \bar{Z})}{(1 + z\bar{z})(1 + Z\bar{Z})} \,, \tag{6.8}$$

the square of the chordal distance between z and Z. Therefore, for the vortex centred at Z,

$$h = \log \frac{(z - Z)(\bar{z} - \bar{Z})}{(1 + z\bar{z})(1 + Z\bar{Z})} + \log 2\epsilon + \epsilon \left(\frac{2}{3} - \frac{(z - Z)(\bar{z} - \bar{Z})}{(1 + z\bar{z})(1 + Z\bar{Z})} \right) + \cdots \,. \tag{6.9}$$

Using this, we can calculate the metric on moduli space. We need the partial derivative with respect to $\bar{z}$ of $h - 2\log|z - Z|$ evaluated at Z. Only the term $-\log(1 + z\bar{z})$ contributes to this, so

$$\begin{aligned} b(Z, \bar{Z}) &= 2\frac{\partial}{\partial \bar{z}}(-\log(1 + z\bar{z}))\Big|_{z=Z} \\ &= -\frac{2Z}{1 + Z\bar{Z}}\,, \end{aligned} \tag{6.10}$$

and hence

$$\frac{\partial b}{\partial Z} = -\frac{2}{(1 + Z\bar{Z})^2}\,. \tag{6.11}$$

The Samols metric (3.3) is therefore, to order ϵ,

$$ds^2\big|_{\mathcal{M}} = \left(\frac{4}{(1 + Z\bar{Z})^2} - (1-\epsilon)\frac{4}{(1 + Z\bar{Z})^2}\right) dZ d\bar{Z} = \frac{4\epsilon\, dZ d\bar{Z}}{(1 + Z\bar{Z})^2}\,. \tag{6.12}$$

This is just a scaled version of the metric on the round sphere where the vortex resides, and the area goes to zero as $\epsilon \to 0$. The result agrees with (3.11), with $R_0 = 2$ and $\frac{1}{\tau} = 1 + \epsilon$.

Let us now consider the vortex on the deformed sphere with conformal factor

$$\Omega = (1 + \alpha P_2(\cos\theta))\Omega_0\,, \tag{6.13}$$

where α is small. This ellipsoidal deformation by the $j = 2$ harmonic $P_2(\cos\theta)$ is axially symmetric in φ. We set $\frac{1}{\tau} = 1+\epsilon$ as before. We work to linear order in α, and in principle to arbitrary order in ϵ (i.e. we consider a vortex of arbitrary size on the slightly deformed sphere). In practice, we shall just calculate up to linear order in ϵ to establish the metric on moduli space close to $\tau = 1$.

Note that the chordal distance (6.8) can be written as

$$2(1 - \mathbf{n}_z \cdot \mathbf{n}_Z)\,, \tag{6.14}$$

where $\mathbf{n}_z = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the unit Cartesian vector corresponding to z, and $\mathbf{n}_Z$ similarly corresponds to Z. If Z is real, we can write $\mathbf{n}_Z = (\sin\lambda, 0, \cos\lambda)$, and then

$$2(1 - \mathbf{n}_z \cdot \mathbf{n}_Z) = 2(1 - \sin\lambda\sin\theta\cos\varphi - \cos\lambda\cos\theta)\,. \tag{6.15}$$

As a function of θ and φ this is a linear combination of $j = 0$ and $j = 1$ harmonics. The deformation of the sphere by a $j = 2$ harmonic will generate products of $j = 1$ and $j = 2$ harmonics. Our calculation of h can proceed because these products are themselves finite sums of harmonics with $j = 1, 2, 3$, by the usual Clebsch–Gordon rules.

The Taubes equation (2.7) on the deformed sphere can be written as

$$\Delta_0 h + (1 + \epsilon - e^h)(1 + \alpha P_2(\cos\theta)) = \frac{4\pi}{\Omega_0}\delta^{(2)}(\mathbf{x} - \mathbf{X})\,. \tag{6.16}$$

We will solve this as before, inverting Δ_0, the Laplacian on the round sphere, on a basis of spherical harmonics. We start again with the case $\mathbf{X} = \mathbf{0}$, and set $h = h_0 + \epsilon h_1 + \cdots$ where h_0 has a $\log \epsilon$ term ensuring that e^h is of order ϵ. h_0 is the solution of eq.(6.16) with the terms $\epsilon - e^h$ dropped but the term proportional to α retained. The solution, this time in trigonometric form, is

$$h_0 = \log \frac{1}{2}(1 - \cos\theta) + \log C_0\epsilon + \frac{\alpha}{6} P_2(\cos\theta)\,, \tag{6.17}$$

with C_0 to be determined. The equation for h_1 is obtained from the terms of order ϵ in (6.16), noting that e^h can be replaced by e^{h_0} at this order. There results the inhomogeneous equation

$$-\Delta_0 h_1 = \left(1 - \frac{C_0}{2}(1 - \cos\theta)\left(1 + \frac{\alpha}{6} P_2(\cos\theta)\right)\right)(1 + \alpha P_2(\cos\theta))\,, \tag{6.18}$$

from which terms quadratic in α are dropped. The right hand side is expressible as a linear combination of $P_m(\cos\theta)$ with $m = 1, 2, 3$, and no constant term, provided $C_0 = 2$ as before. Eq.(6.18) then simplifies to

$$-\Delta_0 h_1 = \left(1 + \frac{7\alpha}{15}\right) P_1(\cos\theta) - \frac{\alpha}{6} P_2(\cos\theta) + \frac{7\alpha}{10} P_3(\cos\theta)\,, \tag{6.19}$$

whose solution is

$$h_1 = \left(\frac{1}{2} + \frac{7\alpha}{30}\right) P_1(\cos\theta) - \frac{\alpha}{36} P_2(\cos\theta) + \frac{7\alpha}{120} P_3(\cos\theta) + C_1\,. \tag{6.20}$$

The constant C_1 can be determined using the equation for h_2.

Now we can tackle the general case, with $\mathbf{X}$ arbitrary. The equation for h_0 differs from the round sphere case by a term of order α and the solution is

$$h_0 = \log \frac{(z - Z)(\bar{z} - \bar{Z})}{(1 + z\bar{z})(1 + Z\bar{Z})} + \log C_0\epsilon + \frac{\alpha}{6} P_2(\cos\theta)\,. \tag{6.21}$$

We recall that

$$\frac{(z - Z)(\bar{z} - \bar{Z})}{(1 + z\bar{z})(1 + Z\bar{Z})} = \frac{1}{2}(1 - \sin\lambda \sin\theta \cos\varphi - \cos\lambda \cos\theta)\,, \tag{6.22}$$

and hence the equation for h_1 is

$$-\Delta_0 h_1 = 1 + \alpha P_2(\cos\theta) - \frac{C_0}{2}(1 - \sin\lambda \sin\theta \cos\varphi - \cos\lambda \cos\theta)\left(1 + \frac{7\alpha}{6} P_2(\cos\theta)\right)\,. \tag{6.23}$$

Multiplying this out, we find yet again that $C_0 = 2$, and then we can express the right hand side in terms of spherical harmonics with $j = 1, 2, 3$. Inverting the Laplacian we find

$$\begin{aligned} h_1 &= \frac{1}{2}\sin\lambda \sin\theta \cos\varphi + \frac{1}{2}\cos\lambda \cos\theta - \frac{\alpha}{36} P_2(\cos\theta) \\ &+ \frac{7\alpha}{6}\cos\lambda \left(\frac{1}{20} P_3(\cos\theta) + \frac{1}{5} P_1(\cos\theta)\right) \\ &+ \frac{7\alpha}{6}\sin\lambda \left(\frac{1}{40} Y_3 - \frac{1}{10}\sin\theta \cos\varphi\right) + C_1 \end{aligned} \tag{6.24}$$

where $Y_3 = (5\cos^2\theta\sin\theta - \sin\theta)\cos\varphi$ is a $j = 3$ harmonic. This can be reorganised in the form

$$h_1 = \frac{1}{2}\Lambda - \frac{\alpha}{36}P_2(\cos\theta) + \frac{7\alpha}{24}\left(\cos\lambda\cos\theta - \frac{1}{2}\Lambda\sin^2\theta\right) + C_1 \tag{6.25}$$

where $\Lambda = \sin\lambda\sin\theta\cos\varphi + \cos\lambda\cos\theta$.

Combining (6.17) and (6.25) gives $h = h_0 + \epsilon h_1 + \dots$ to order ϵ. Next we convert h to a function of z and $\bar{z}$, using (6.22) and $\cos\theta = \frac{1-z\bar{z}}{1+z\bar{z}}$. We then remove the logarithmically singular term, defining $\tilde{h} = h - 2\log|z - Z|$, and calculate the Samols coefficient

$$b = 2\frac{\partial}{\partial\bar{z}}\tilde{h}\Big|_{z=Z} . \tag{6.26}$$

This simplifies to

$$b = -\frac{2Z}{1 + Z\bar{Z}} - 2\alpha(1+\epsilon)\frac{(1 - Z\bar{Z})Z}{(1 + Z\bar{Z})^3} , \tag{6.27}$$

where we have used $\cos\lambda = \frac{1-Z\bar{Z}}{1+Z\bar{Z}}$. The term proportional to α is new here. Expression (6.27) has been obtained assuming Z is real, but by axial symmetry it is valid for all Z. Differentiating again, we find

$$\frac{\partial b}{\partial Z} = -\frac{2}{(1 + Z\bar{Z})^2} - 2\alpha(1+\epsilon)\frac{1 - 4Z\bar{Z} + (Z\bar{Z})^2}{(1 + Z\bar{Z})^4} , \tag{6.28}$$

where the final expression is equivalently

$$\frac{1 - 4Z\bar{Z} + (Z\bar{Z})^2}{(1 + Z\bar{Z})^4} = \frac{P_2(\cos\lambda)}{(1 + Z\bar{Z})^2} . \tag{6.29}$$

Finally, we calculate the metric on moduli space $\mathcal{M}$,

$$ds^2\big|_{\mathcal{M}} = \left(\Omega + 2\tau\frac{\partial b}{\partial Z}\right) dZ d\bar{Z} , \tag{6.30}$$

using $\Omega = (1 + \alpha P_2(\cos\lambda))\Omega_0$ and the approximation $\tau = 1 - \epsilon$. The terms not proportional to ϵ cancel, as do the terms involving $P_2(\cos\lambda)$, leaving

$$ds^2\big|_{\mathcal{M}} = \frac{4\epsilon\, dZ d\bar{Z}}{(1 + Z\bar{Z})^2} \tag{6.31}$$

with order ϵ^2 corrections which could be found by systematically extending this calculation. As expected, the area of the moduli space is ϵ times the area of Σ, and vanishes as $\epsilon \to 0$. Remarkably, the deformation of the sphere also vanishes from the moduli space at this order. These conclusions are in agreement with what occurs in Ricci flow. Further calculation might show a deformation relative to the round sphere of order ϵ, although agreement with Ricci flow would require a deformation only at order ϵ^2.

7 Vortex on a torus

On a surface of genus 1, a torus, Ricci flow and normalised Ricci flow are the same. Given any starting metric, Ricci flow takes it to a flat metric, while preserving the conformal class and the area of the torus. From the linearized Ricci flow equation, one sees that the approach to the flat metric takes infinite time.

Consider now one vortex on a torus Σ with local complex coordinate $z = x + iy$. We simplify our analysis by choosing a square torus, $\{(x, y) : 0 \leq x \leq 1\,, 0 \leq y \leq 1\}$ with opposite sides identified. The metric is $ds^2 = \Omega\, dz d\bar{z}$ with Ω a smooth function on the torus, and we assume

$$\int_0^1 \int_0^1 \Omega\, dxdy = 4\pi\,, \tag{7.1}$$

so the area of the torus is 4π. The Higgs field and gauge potential are a section and connection of a $U(1)$ bundle with Chern number 1. We take the bundle to be defined by the transition function $g(x) = e^{-2\pi i x}$ relating the top edge ($y = 1$) to the bottom edge ($y = 0$), and by the trivial transition function relating the right edge ($x = 1$) to the left edge ($x = 0$).

As always, the 1-vortex moduli space $\mathcal{M}$ is a copy of Σ, with the vortex centre Z as coordinate, and metric

$$ds^2\big|_{\mathcal{M}} = \Omega(\tau)\, dZ d\bar{Z} = \left(\Omega + 2\tau \frac{\partial b}{\partial Z}\right) dZ d\bar{Z}\,, \tag{7.2}$$

with notation as before. The cotangent bundle of the torus is trivial, so b is a well-defined, smooth function on $\mathcal{M}$.

Our earlier discussion shows that for small τ, $\Omega(\tau)$ evolves from the initial metric Ω according to the Ricci flow. Moreover, it follows from (7.2) that for all τ, $\mathcal{M}$ is conformally a square torus and also, since the Chern number of the cotangent bundle vanishes, that the area of $\mathcal{M}$ is 4π, the same as the area of Σ. We shall show in this section that in the Bradlow limit $\tau \to 1$, $\Omega(\tau)$ becomes the flat metric. All this is similar to the Ricci flow, with one important difference. $\Omega(\tau)$ is defined only on the finite interval $0 < \tau < 1$. The time t in the Ricci flow, and the Bradlow parameter τ, cannot therefore be identified.

It is easiest to understand what happens in the strict Bradlow limit, $\tau = 1$. The moduli space does not collapse to a point here, as it did for a vortex on a sphere. The Bogomolny equations reduce to

$$B = \frac{1}{2}\Omega\,, \tag{7.3}$$

and ϕ vanishes everywhere. However, the connection is not completely determined by (7.3). Given a 1-form gauge potential $a^{(0)}$, the general one is $a = a^{(0)} + \alpha$ where α has zero magnetic field, i.e. α is a flat connection on Σ. By a gauge choice we can assume

$$\alpha = 2\pi(\mu dx + \nu dy) \tag{7.4}$$

where μ and ν are independent of x and y. Furthermore, we can restrict μ, ν to the ranges $-\frac{1}{2} \leq \mu \leq \frac{1}{2}$, $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$ (with endpoints identified), since a legitimate gauge transformation on the torus is $g(x,y) = e^{2\pi i(mx+ny)}$ with m, n integers, and this shifts μ by m and ν by n. The moduli space $\mathcal{M}$ at $\tau = 1$ is therefore the square torus $\{(\mu,\nu) : -\frac{1}{2} \leq \mu \leq \frac{1}{2}, -\frac{1}{2} \leq \nu \leq \frac{1}{2}\}$. The metric on $\mathcal{M}$ can be directly obtained from the kinetic part of the Lagrangian for time-dependent μ and ν. This is

$$T = \frac{1}{2}\int_0^1 \int_0^1 (e_x^2 + e_y^2)\, dxdy\,, \tag{7.5}$$

where

$$e_x = \dot{a}_x - \partial_x a_0\,, \quad e_y = \dot{a}_y - \partial_y a_0 \tag{7.6}$$

are the electric field components, with a_0 the time component of the gauge potential. Note that factors of Ω cancel out in T. In our gauge, it is consistent with Gauss' law, $\partial_x e_x + \partial_y e_y = 0$, to set $a_0 = 0$, so $e_x = 2\pi\dot{\mu}$ and $e_y = 2\pi\dot{\nu}$, and T reduces to

$$T = 2\pi^2(\dot{\mu}^2 + \dot{\nu}^2)\,. \tag{7.7}$$

The metric on moduli space is therefore

$$ds^2\big|_{\mathcal{M}} = 4\pi(d\mu^2 + d\nu^2)\,, \tag{7.8}$$

where we have extracted the vortex mass π in the limit $\tau = 1$ (the energy in the magnetic field and Higgs field) and the factor $\frac{1}{2}$. The metric is clearly flat, whatever Ω was initially, and the area of $\mathcal{M}$ is 4π as expected.

Before moving on, it is convenient to note here how the moduli of the flat connection can be described in an alternative gauge. Suppose we perform the gauge transformation $g(x,y) = e^{-2\pi i(\mu x + \nu y)}$. α now vanishes, but instead, the bundle transition functions are $g(x) = e^{-2\pi i(x+\nu)}$ relating $y = 1$ to $y = 0$, and $g(y) = e^{-2\pi i\mu}$ relating $x = 1$ to $x = 0$. For time-dependent μ and ν, $a_0 = -2\pi(\dot{\mu}x + \dot{\nu}y)$, consistent with a_0 on opposite edges of the square being related by the time derivative of the transition functions, and the electric field is as before.

It is fairly clear that the metric on $\mathcal{M}$ smoothly approaches the limiting flat metric (7.8) as τ approaches 1. This is because the magnetic field approaches the limiting value $\frac{\Omega}{2}$, and the Higgs field becomes vanishing small. However, what needs clarification is how the time-varying moduli μ and ν are related to the actual motion of the vortex, that is, the motion of its centre Z. We shall now show that Z is linearly related to μ and ν when τ is close to 1, and that the moduli space $\mathcal{M}$ has the canonical flat metric with Z as complex coordinate. We shall calculate this metric just at zeroth order in $\epsilon = \frac{1}{\tau} - 1$. Our method is rather close to that of refs.[8, 9].

At zeroth order, $B = \frac{\Omega}{2}$. We rewrite this as

$$B = 2\pi + \frac{\tilde{\Omega}}{2} \tag{7.9}$$

where $\tilde{\Omega}$ integrates to zero. Then one choice of gauge potential is

$$a_x = -2\pi y - \frac{1}{2}\partial_y \tilde{K}\,, \quad a_y = \frac{1}{2}\partial_x \tilde{K}\,, \tag{7.10}$$

where $\nabla^2 \tilde{K} = \tilde{\Omega}$. This Poisson equation has a unique solution for $\tilde{K}$ on the torus, up to an irrelevant additive constant. The gauge potential (7.10) is consistent with either of the gauge choices we introduced above, and we choose the second of these, with the moduli μ and ν present in the transition functions.

The moduli are physical, since they affect the holonomy of the gauge potential (7.10) around cycles in the x- and y-directions. More importantly for us, they affect the Higgs field. Recall the first Bogomolny equation

$$D_{\bar{z}}\phi \equiv \frac{1}{2}(\partial_x + i\partial_y - ia_x + a_y)\phi = 0\,. \tag{7.11}$$

We need to solve this with the above gauge potential, and boundary conditions

$$\begin{aligned} \phi(x, y = 1) &= \phi(x, y = 0)e^{-2\pi i(x+\nu)}\,, \\ \phi(x = 1, y) &= \phi(x = 0, y)e^{-2\pi i\mu}\,. \end{aligned} \tag{7.12}$$

We may use the integrating factor $K = \pi y^2 + \frac{1}{2}\tilde{K}$, and set $\phi = e^{-K}\psi$. Then (7.11) reduces to $\partial_{\bar{z}}\psi = 0$, so $\psi(z)$ is holomorphic. The boundary conditions (7.12) become

$$\begin{aligned} \psi(z + 1) &= \psi(z)e^{-2\pi i\mu}\,, \\ \psi(z + i) &= \psi(z)e^{-2\pi iz}e^{\pi}e^{-2\pi i\nu}\,. \end{aligned} \tag{7.13}$$

These are exactly the defining equations for a theta-function with characteristics on a square torus [5], so

$$\psi(z) = C\Theta_{\left[\substack{-2\mu \\ 2\nu}\right]}(z; i)\,. \tag{7.14}$$

C is a normalisation constant depending on Z and $\bar{Z}$. Its phase is arbitrary, but $|C|^2$ is determined by eq.(2.8) and is of order ϵ. The theta-function has one zero in the unit square, at

$$Z = \frac{1 - 2\nu}{2} + \frac{1 + 2\mu}{2}\,i \tag{7.15}$$

and this is the vortex centre. So, Z is linearly related to μ and ν, and if Z moves at constant velocity, there is a constant electric field orthogonal to the velocity.

Now we can use Samols' formula to find the metric on moduli space. As usual, we need to expand $h = \log|\phi|^2$ around Z. We set $z = Z + w$ and use the identity

$$\begin{aligned} &\Theta_{\left[\substack{-2\mu \\ 2\nu}\right]}\left(\frac{1 - 2\nu}{2} + \frac{1 + 2\mu}{2}i + w; i\right) \\ &\qquad = e^{2\pi i\left[-\frac{1}{2}(1+2\mu)w - \frac{1}{8}(1+2\mu)^2 i - \frac{1}{4}(1+2\mu)\right]}\Theta_{\left[\substack{1 \\ 1}\right]}(w; i)\,, \end{aligned} \tag{7.16}$$

where the final theta-function has the expansion around $w = 0$,

$$\Theta_{\left[\substack{1\\1}\right]}(w; i) = \alpha w(1 + \gamma w^2 + \cdots) . \tag{7.17}$$

Therefore,

$$h = -2\pi y^2 - \tilde{K} + 2\log|C| + \frac{\pi}{2}(1+2\mu)^2 + 2\pi(1+2\mu)\mathrm{Im}\, w + 2\log|\alpha| + 2\log|w| + \cdots . \tag{7.18}$$

Since $y = \frac{1+2\mu}{2} + \mathrm{Im}\, w$, this expansion of h simplifies to the desired form

$$h = 2\log|w| + a + \frac{1}{2}\bar{b}w + \frac{1}{2}b\bar{w} + \cdots \tag{7.19}$$

where

$$b = -2\partial_{\bar{w}}\tilde{K}\Big|_{w=0} = -2\partial_{\bar{z}}\tilde{K}\Big|_{z=Z} . \tag{7.20}$$

Therefore, using the Samols formula (7.2),

$$ds^2\Big|_{\mathcal{M}} = (\Omega - \tau\nabla^2\tilde{K})\, dZd\bar{Z} , \tag{7.21}$$

with Ω and $\tilde{K}$ here regarded as functions of Z and $\bar{Z}$. But recall that $\nabla^2\tilde{K} = \Omega - 4\pi$, and $\tau \simeq 1 - \epsilon$, so to zeroth order in ϵ,

$$ds^2\Big|_{\mathcal{M}} = 4\pi\, dZd\bar{Z} . \tag{7.22}$$

This result in the case that Σ is a flat torus has been obtained previously. What is novel is that in the Bradlow limit, to zeroth order in ϵ, the contribution of $\tilde{K}$, which accounts for the deformation of the metric on Σ, cancels out in the metric on the moduli space. Calculating the metric on $\mathcal{M}$ to order ϵ, for any non-trivial $\tilde{K}$, would be a tricky exercise in theta-functions, and we have not attempted it.

8 Conclusion

For one abelian Higgs vortex on a compact surface Σ with arbitrary metric, we have studied the metric on the moduli space $\mathcal{M}$. The metric on $\mathcal{M}$ is in the same conformal class as the metric on Σ, and it has been shown to exhibit an interesting geometrical evolution as the Bradlow parameter τ varies. For small τ, the metric evolves from the original metric on Σ in the same way as in Ricci flow. Further similarities to Ricci flow have been established by studying a vortex on a slightly deformed sphere, and on a deformed square torus, as τ approaches the Bradlow limit (which is $\tau = 1$ if Σ has area 4π). Possibly there is a precise relation for all τ between the geometrical flow of the moduli space metric and Ricci flow, although the example of a vortex on a torus shows that there must be at least a difference in the time parametrization. It would be interesting to understand the metric on the moduli space in the Bradlow limit in the case that the genus g of Σ is greater than 1, since this metric appears

to depend only on the conformal class of Σ. Nasir has studied the metric for N-vortices with $N \geq g$ in the Bradlow limit, but did not obtain an explicit result in the 1-vortex case [18].

Since vortex motion is approximately along a geodesic in moduli space, our results suggest that the metrics generated by Ricci flow have a physical interpretation as effective metrics whose geodesics model the motion of particles of finite size. These metrics replace the starting metric whose geodesics model the motion of pointlike particles.

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